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(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FOURTH SEMESTER EXAMINATION, MAY 2015

SECOND YEAR

MATHEMATICS (Honours)

Date : 22/05/2015

Time : 11 am – 3 pm

Paper : IV

Full Marks : 100

[Use a separate Answer book for each group]

Group – A

Unit - I

[*Answer* ***any five*** *questions*]

1. a) Let $X=\{\{x_n\}: x_n \in \mathbb{R}\ \forall\ n\in\mathbb{N}\}$. For any two points x = $\{x_n\}$ and y = $\{y_n\}$ of X define

$$d: X\times X\to\mathbb{R} \text{ by } d(x,y)=\begin{cases} 0, & \text{if } x=y \\ \dfrac{1}{\alpha(x,y)}, & \text{if } x\neq y \end{cases}$$

where $\alpha(x,y)=\min\{n\in\mathbb{N}: x_n\neq y_n\}$. Show that (X,d) is a metric space. [5]

b) Show that the set $\{0,1\}$ is a G_δ subset of $\mathbb{R}$. [2]

2. a) Define a separable metric space. Prove that a separable metric space is 2^{nd} countable. [5]

b) Let 'd' be a metric on $\mathbb{N}$. Is $(\mathbb{N},d)$ 2^{nd} countable? Justify your answer. [2]

3. a) Let X be a complete metric space. Prove that for any decreasing sequence $\{F_n\}$ of nonempty closed sets in X with diam $(F_n)\to 0$ as $n\to\infty$, the set $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. [4]

b) In 3(a) if $\{F_n\}$ is a decreasing sequence of nonempty bounded closed sets but $\{\text{diam}(F_n)\}$ is not a null sequence do you think that $\bigcap_{n=1}^{\infty} F_n$ contains more than one point? Justify your answer. [3]

4. a) Let $X\subseteq\mathbb{R}$ be such that each continuous function $f: X\to\mathbb{R}$ is bounded. Show that X is compact. [4]

b) Let A, B be two disjoint closed sets in a metric space X. Construct a continuous function $f: X\to\mathbb{R}$ with $f(A)=\{0\}$, $f(B)=\{1\}$. [3]

5. a) Let X be a metric space such that each continuous map $f: X\to\mathbb{R}$ be uniformly continuous. Show that X is complete. [4]

b) Let $f, g: X\to Y$ be two continuous maps, where X and Y are two metric spaces. Show that the set $\{x\in X: f(x)=g(x)\}$ is a closed set in X. [3]

6. a) Show that if X is a sequentially compact metric space then every open cover of X has a Lebesgue number. [4]

b) Test the uniform continuity of the function $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\sqrt{x}$. [3]

7. Let $X=\{\{x_n\} \mid x_n\in\mathbb{R}\ \forall n\ \&\sum|x_n|<\infty\}$. For each $n\in\mathbb{N}$ define $\{e_n\}\in X$ such that nth term of the sequence $\{e_n\}$ is 1 and all other terms are zero. Consider $E=\{\{e_n\} \mid n\in\mathbb{N}\}$ and consider the metric 'd' defined on X by $d(\{x_n\},\{y_n\})=\sum_{n=1}^{\infty}|x_n-y_n|$. Show that

a) E is closed and bounded subset of X. [4]

b) Is E a compact subset of X? Justify your answer. [3]

8. a) Prove that a connected metric space with at least two distinct points is uncountable. [3]

b) Let $f, g: [0,1] \to [0,1]$ be two continuous functions such that g(0) = 0 & g(1) = 1. Prove that there exists $x \in [0,1]$ such that f(x) = g(x). [4]

Unit - II

[*Answer **any three** questions*]

9. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \begin{cases} nx & : \ 0 \le x \le 1/n \\ 2 - nx & : \ 1/n \le x \le 2/n \\ 0 & : \ 2/n \le x \le 1 \end{cases}$

a) If the sequence of function $(f_n)_{n\ge1}$ converges pointwise to a function f then find 'f'. [2]

b) Do your think that $(f_n)_{n\ge1}$ converges to f uniformly? Justify your answer. [3]

10. In a power series $\sum_{n=0}^{\infty} a_n x^n$ be neither no where convergent nor every where convergent, then show that there exists a real number R(>0) such that the series converges absolutely for all x satisfying $|x| < R$ and diverges for all x satisfying $|x| > R$. [5]

11. a) Let $\{f_n\}$ be a sequence of functions converging uniformly on [a,c] and [c,b], where a<c<b. Then prove that $\{f_n\}$ converges uniformly on [a,b]. [2]

b) For each $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} nx^2 & , \ 0 \le x \le 1/n \\ x & , \ 1/n < x \le 1 \end{cases}$

Then show that $\{f_n\}$ is uniformly convergent on [0,1]. [3]

12. a) Prove that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1} x^n}{n^p (1+x^n)}$ is uniformly convergent for all p > 0 on [0, 1]. [3]

b) Show that the series $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n+x^2}$ is uniformly convergent for all real x but not absolutely convergent for any real x. [2]

13. a) Find the radius of convergence of the following power series :

$$1 - \frac{2^2}{3^2} x + \frac{2^2.4^2}{3^2.5^2} x^2 - \frac{2^2.4^2.6^2}{3^2.5^2.7^2} x^3 + \ldots$$ [3]

b) If the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R then what the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n^2 x^n$? Justify your answer. [2]

Group – B

Unit - I

[*Answer **any three** questions*]

14. a) Solve the equation $3x^2 \frac{d^2y}{dx^2} + (2 + 6x - 6x^2) \frac{dy}{dx} - 4y = 0$ by factorisation of operators. [4]

b) Find the eigen values λ_n and eigen functions $y_n(x)$ for the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \lambda y = 0, \ (\lambda > 0).$$ [4]

c) Use the convolution theorem to find $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$. [2]

15. a) Solve $(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x(1-x^2)$, given that y = x is a solution of its reduced equation. [5]

b) Apply Charpit's method to find the complete integral of the equation $(zp+x)^2+(zq+x)^2=1$ where $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}$. [5]

16. a) Show that the equation $(y^2+z^2-x^2)dx-2xydy-2zxdz=0$ is integrable. Solve the equation. [2+3]

b) Solve the equation $(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-xy=0$ in series, near the point x = 0. [5]

17. a) Write down the conditions for existence of Laplace transform of a function F(t), $t\geq 0$. Cite an example to show that the conditions stated for existence are not necessary. [2+3]

b) Solve the simultaneous equation $\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$. [5]

18. a) Find the equation of the integral surface of the linear differential equation $2y(z-3)p+(2x-z)q=y(2x-3)$ which passes through the circle $x^2+y^2=2x, z=0$. [5]

b) Solve the differential equation $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4e^{2t}; y(0)=-3, y'(0)=5$, with the help of Laplace transform. [5]

Unit - II

[*Answer **any four** questions*]

19. a) Show that the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to a focus is $\frac{b^2}{p^2}=\frac{2a}{r}-1$. [3]

b) Test for the existence of asymptotes of the curve $y=x\frac{x^2+a^2}{x^2-a^2}$. [2]

20. Find the asymptotes, if any, of the curve $y=2\log\sec\left(\frac{x}{2}\right)$. [5]

21. a) Obtain the pedal of $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ with respect to the origin. [3]

b) Investigate the curve $y^2-x^4+x^6=0$ for existence of double points. [2]

22. Find the envelope of the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ where ab = 1, 'a' and 'b' being variable parameters. [5]

23. Show that the radius of curvature of the curve $x=ae^{\theta}(\sin\theta-\cos\theta)$, $y=ae^{\theta}(\sin\theta+\cos\theta)$ and its evolute at corresponding point are equal. [5]

24. a) Find the envelope of the lines $\frac{x}{\sqrt{\sin\theta}}+\frac{y}{\sqrt{\cos\theta}}=a$. [3]

b) Find the area of the region bounded by the curve $y=x(x-1)(x-2)$ and the x-axis. [2]

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